This file is just a bunch of random stuff and notes for myself (and others, of course).

1 ENU to NED transformations

I had the problem very often that I have to transform form ENU no NED. The simple conversion: “Flip x and y and negate z” doesn’t work for quaternions or if you want to use matrix algebra.

1.1 Matrix

Flipping x and y and negating z is easy to express as a matrix:

$$R_{ENU2NED} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{1}$$

This works in both directions, since $R_{ENU2NED} = R^T_{ENU2NED}$.

1.2 Quaternion

It’s easy to compute a quaternion out of the above rotation matrix.

$$q_{ENU2NED} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(i + j) \tag{2}$$

This makes sense, since the real value $= 0$ represents a rotation about $180°$ and the three values for the axis $\overrightarrow{v} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T$ represent the axis of rotation.

Transforming a quaternion between ENU/NED

If you want to cahnge a quaternion from NED to ENU or vice versa. It’s not totally simple like for vectors.
If your quaternion consist of the values:

$$q_{ECEF2NED} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \tag{3}$$

Then a transformation to ENU (NED) is made as following:

$$q_{ECEF2NED} = \frac{1}{\sqrt{2}} \begin{pmatrix} -b - c \\ a + d \\ a - d \\ -b + c \end{pmatrix} \tag{4}$$

Pay attention doing it twice! The multiplication of an NED to ENU quaternion with itself leads to

$$q_{ECEF2NED} \bullet q_{ECEF2NED} \bullet \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -a \\ -b \\ -c \\ -d \end{pmatrix}. \tag{5}$$

This is logically the same rotation, but mathemaically a different quaternion. So don’t be confused if all values are negative :-)

2 Initialisation

2.1 What about the standard deviation?

Pay attention to the following equations! Many things are just guesses and no really mathematically proofen. So better think twice before just using my assumptions.

$$\begin{tikzcd}
\text{Noise} \arrow[d, "\text{Gaussian noise}"] \\
\text{Measurement} \arrow[d, "\text{weight of a single measurement}"] \\
\text{attitude profile matrix} \arrow[d, "\text{(see the section below)}"] \\
\text{“K” - matrix} \arrow[d, "\text{computational error}"] \\
\text{Eigenvector} \arrow[d, "\text{computational error}"] \\
\text{Euler angle}
\end{tikzcd}$$

Fig. 1: Propagation of uncertainty

Measurement

First of all, every sensor (accelerometers $\overrightarrow{a}$ and magnetometers $\overrightarrow{m}$) has gaussian noise, that can be expressed as an additive error:

$$\overrightarrow{a} + \overrightarrow{\sigma_a} \qquad \overrightarrow{m} + \overrightarrow{\sigma_m} \tag{6}$$

It can be asssumed that the error follows a standard deviation (has zero mean and is time-invariant).

attitude profile matrix

The attitude profile matrix $\mathbf{B}$ is the sum of the measurements with specific weights.

$$\mathbf{B} = \sum_{k=1}^{n} w_k \cdot \overrightarrow{W}_k \cdot \overrightarrow{V}_k^T = w_a \sum_{k=1}^{n_a} \overrightarrow{a}_k \cdot \overrightarrow{g}^T + w_m \sum_{k=1}^{n_m} \overrightarrow{m}_k \cdot \overrightarrow{h}^T \tag{7}$$

n is the number of measurements, w_k is the specific weight of a measurement, $\overrightarrow{W}_k$ the measured vector and $\overrightarrow{V}_k$ the reference direction, which belongs to the measured direction. Therefore n_a is the number of acceleration measurements, w_a is the (constant) weight of the acceleration measurements, $\overrightarrow{a}_k$ is a single acceleration observation and $\overrightarrow{g}$ is the gravity. $\overrightarrow{a}_k$ becomes normed. Similar for the magnetometer weight w_m, measurement $\overrightarrow{m}_k$, the magnetic field $\overrightarrow{h}$ and the amount of magnetometer measurements n_m. See the next section how the weight should be choosen.

The resulting error is

$$\sigma_{\mathbf{B}} = \frac{n_a}{f_a}\frac{1}{\|g\|_2}\overrightarrow{\sigma_a}\overrightarrow{g}^T + \frac{n_m}{f_m}\frac{1}{\|h\|_2}\overrightarrow{\sigma_m}\overrightarrow{m}^T \tag{8}$$

"K"-matrix

The error for the "K"-matrix is easy to get by inserting $\mathbf{B} + \sigma_{\mathbf{B}}$ into

$$\mathbf{K} = \begin{bmatrix} trace(\mathbf{B}) & \overrightarrow{Z}^T \\ \overrightarrow{Z} & \mathbf{B} + \mathbf{B}^T - trace(\mathbf{B})\mathbf{I} \end{bmatrix} \tag{9}$$

$$\sigma_{\mathbf{K}} = \begin{bmatrix} trace(\sigma_{\mathbf{B}}) & \overrightarrow{\sigma}_Z^T \\ \overrightarrow{\sigma}_Z & \sigma_{\mathbf{B}} + \sigma_{\mathbf{B}}^T - trace(\sigma_{\mathbf{B}})\mathbf{I} \end{bmatrix} \tag{10}$$

Eigenvector

The dominant eigenvector is computed with the power iteration:

$$\overrightarrow{x}_{k+1} = \frac{\mathbf{K}\overrightarrow{x}_k}{\|\mathbf{K}\overrightarrow{x}_k\|_{max}} \tag{11}$$

With $\mathbf{K} \to \mathbf{K} + \sigma_{\mathbf{K}}$:

$$\overrightarrow{x}_{k+1} + \overrightarrow{\sigma}_{x_{k+1}} = \frac{(\mathbf{K} + \sigma_{\mathbf{K}})\overrightarrow{x}_k}{\|(\mathbf{K} + \sigma_{\mathbf{K}})\overrightarrow{x}_k\|_{max}} = \frac{\mathbf{K}\overrightarrow{x}_k}{\|(\mathbf{K} + \sigma_{\mathbf{K}})\overrightarrow{x}_k\|_{max}} + \frac{\sigma_{\mathbf{K}}\overrightarrow{x}_k}{\|(\mathbf{K} + \sigma_{\mathbf{K}})\overrightarrow{x}_k\|_{max}} \tag{12}$$

If we assume that

$$\|(\mathbf{K} + \sigma_{\mathbf{K}})\overrightarrow{x}_k\|_{max} \approx \|\mathbf{K}\overrightarrow{x}_k\|_{max} \tag{13}$$

we'll get:

$$\overrightarrow{\sigma}_{x_{k+1}} = \frac{\sigma_{\mathbf{K}}\overrightarrow{x}_k}{\|\mathbf{K}\overrightarrow{x}_k\|_{max}} \tag{14}$$

This means that our final error at the end of the iteration can be computed using the vector from the step before:

$$\overrightarrow{\sigma}_{x_n} = \frac{\sigma_{\mathbf{K}}\overrightarrow{x}_{n-1}}{\|\mathbf{K}\overrightarrow{x}_{n-1}\|_{max}} \tag{15}$$

We should keep in mind, that we get an additional error because the the power iteration does not stop, when it's close to the eigenvector. It stops if two iteration steps $\overrightarrow{x}_k$ and $\overrightarrow{x}_{k+1}$ are close to each other. To avoid getting an additional problem with that we should choose the canceling condition of the iteration $\delta = \|\overrightarrow{x}_k - \overrightarrow{x}_{k+1}\|_{max}$ much smaller than $\|\sigma_{\mathbf{K}}\|_{max}$.

And, of course:

$$\overrightarrow{\sigma}_x = q_\sigma \tag{16}$$

Euler angles

Our current expression is something like $q_{true} = q_{false} + \sigma_q$. But it would be helpful to express the difference as a multiplication:

$$q_{false} + \sigma_q = q_{false2true} \bullet q_{false} \tag{17}$$
$$(q + \sigma_q) \bullet q_{false}^{-1} = q_{false2true} \bullet q_{false} \bullet q_{false}^{-1} \tag{18}$$
$$q_{false2true} = (q_{false} + \sigma_q) \bullet q_{false}^{-1} \tag{19}$$
$$q_{false2true} = \mathbf{I} + \sigma_q \bullet q_{false}^{-1} \tag{20}$$
$$q_{false2true} = \mathbf{I} + \sigma_q \bullet q_{false}^{*} \tag{21}$$

Composition of small Euler angles[1] is done by simply adding them. We can assume that the error of the quaternion is small, so addition should be valid.

$$q_{false2true} \bullet q_{false} \rightarrow \begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix} + \begin{pmatrix} \sigma_\phi \\ \sigma_\theta \\ \sigma_\psi \end{pmatrix} q_{false2true} \rightarrow \begin{pmatrix} \sigma_\phi \\ \sigma_\theta \\ \sigma_\psi \end{pmatrix} \tag{22}$$

For small rotations, the imaginary part $\overrightarrow{v}$ of the quaternion becomes Euler angles.

$$\overrightarrow{v}_{false} = \begin{pmatrix} \sigma_\phi \\ \sigma_\theta \\ \sigma_\psi \end{pmatrix} \tag{23}$$

As a result, we don't care about the Identity rotation from equation (21). Furthermore, with

$$q_{false} = \begin{pmatrix} q_0 \\ \overrightarrow{v}_q \end{pmatrix} \quad and \quad \sigma_q = \begin{pmatrix} \sigma_{q0} \\ \overrightarrow{v}_\sigma \end{pmatrix} \tag{24}$$

we can rewrite equation (23) to

$$\begin{pmatrix} \sigma_\phi \\ \sigma_\theta \\ \sigma_\psi \end{pmatrix} = q_0 \overrightarrow{v}_\sigma - \sigma_{q0} \overrightarrow{v}_q + \overrightarrow{v}_q \times \overrightarrow{v}_\sigma . \tag{25}$$

2.2 choosing the best weight for the attitude profile matrix

If you replace the single measurements in equation (7) with the real measurements

$$\overrightarrow{a}_k + \overrightarrow{\sigma_a} \qquad \overrightarrow{m}_k + \overrightarrow{\sigma_m} \tag{26}$$

and assume that $\mathbf{B}$ has an error $\mathbf{B} + \sigma_{\mathbf{B}}$, you will get

$$\mathbf{B} + \sigma_{\mathbf{B}} = w_a \sum_{k=1}^{n_a} (\overrightarrow{a}_k + \overrightarrow{\sigma_a}) \cdot \overrightarrow{g}^T + w_m \sum_{k=1}^{n_m} (\overrightarrow{m}_k + \overrightarrow{\sigma_m}) \cdot \overrightarrow{h}^T \tag{27}$$

$$\mathbf{B} + \sigma_{\mathbf{B}} = w_a \sum_{k=1}^{n_a} \overrightarrow{a}_k \cdot \overrightarrow{g}^T + \overrightarrow{\sigma_a} \cdot \overrightarrow{g}^T + w_m \sum_{k=1}^{n_m} \overrightarrow{m}_k \cdot \overrightarrow{h}^T + \overrightarrow{\sigma_m} \cdot \overrightarrow{h}^T \tag{28}$$

$$\mathbf{B} + \sigma_{\mathbf{B}} = \underbrace{w_a \sum_{k=1}^{n_a} \overrightarrow{a}_k \cdot \overrightarrow{g}^T + w_m \sum_{k=1}^{n_m} \overrightarrow{m}_k \cdot \overrightarrow{h}^T}_{\mathbf{B}} + w_a \sum_{k=1}^{n_a} \overrightarrow{\sigma_a} \cdot \overrightarrow{g}^T + w_m \sum_{k=1}^{n_m} \overrightarrow{\sigma_m} \cdot \overrightarrow{h}^T \tag{29}$$

[1] *only* small Euler angles!

$$\sigma_{\mathbf{B}} = w_a \sum_{k=1}^{n_a} \overrightarrow{\sigma_a} \cdot \overrightarrow{g}^T + w_m \sum_{k=1}^{n_m} \overrightarrow{\sigma_m} \cdot \overrightarrow{h}^T \tag{30}$$

The sums are independent from their indices:

$$\sigma_{\mathbf{B}} = w_a n_a \overrightarrow{\sigma_a} \cdot \overrightarrow{g}^T + w_m n_m \overrightarrow{\sigma_m} \cdot \overrightarrow{h}^T \tag{31}$$

It would be nice, if it's possible to reduce this to a single value. To do that, we need a matrix norm. In this case, I choosed the Frobenius Norm:

$$\begin{aligned}
\|\sigma_{\mathbf{B}}\|_F &= \|w_a n_a \overrightarrow{\sigma_a} \cdot \overrightarrow{g}^T + w_m n_m \overrightarrow{\sigma_m} \cdot \overrightarrow{h}^T\|_F && (32)\\
&\leq \|w_a n_a \overrightarrow{\sigma_a} \cdot \overrightarrow{g}^T\|_F + \|w_m n_m \overrightarrow{\sigma_m} \cdot \overrightarrow{h}^T\|_F && (33)\\
&= w_a n_a \|\overrightarrow{\sigma_a} \overrightarrow{g}^T\|_F + w_m n_m \|\overrightarrow{\sigma_m} \overrightarrow{h}^T\|_F && (34)
\end{aligned}$$

It is straight-forward to proove that $\|\overrightarrow{a} \overrightarrow{b}^T\|_F = \|a\|_2 \cdot \|b\|_2$

$$\|\sigma_{\mathbf{B}}\|_F \leq w_a n_a \|g\|_2 \cdot \|\sigma_a\|_2 + w_m n_m \|h\|_2 \cdot \|\sigma_m\|_2 \tag{35}$$

As you can see, the uncertainty depends on the following parameters:

- The weight of a measurement w_a and w_m.
- The number of measurements n_a and n_m.
- The length/norm of the reference directions $\overrightarrow{g}$ and $\overrightarrow{h}$.
- The maximum of the error σ_a and σ_m.

This is not what I want. I don't want the error grow with the number of measurements or with the length of the reference direction, which is related to the kind of the measurement. If I choose

$$w_a = \frac{1}{n_a \cdot \|g\|_2} \quad \mathit{and} \quad w_m = \frac{1}{n_m \cdot \|h\|_2} \tag{36}$$

I get something like

$$\|\sigma_{\mathbf{B}}\|_F \leq \|\sigma_a\|_2 + \|\sigma_m\|_2 \quad , \tag{37}$$

which looks much better. For the Frobenius norm of the attitude profile matrix the choosen weight leads to

$$\|\mathbf{B}\|_F \leq \frac{1}{n_a} \sum_{k=1}^{n_a} \|a_k\|_2 + \frac{1}{n_m} \sum_{k=1}^{n_m} \|m_k\|_2 \quad . \tag{38}$$

That is an acceptable fact, since it helps to keep the matrix bound. But because I want to do live-update of the attitude profile matrix I don't know the real amount of measurements n_a and n_m. But I know the measurement frequencies f_a and f_m, which are directly linked to them ($f = \frac{n}{T}$). So my final decision for the measurement weight is

$$w_a = \frac{1}{f_a \cdot \|g\|_2} \quad \mathit{and} \quad w_m = \frac{1}{f_m \cdot \|h\|_2} \quad . \tag{39}$$

The resulting error is then

$$\sigma_{\mathbf{B}} = \frac{n_a}{f_a} \frac{1}{\|g\|_2} \overrightarrow{\sigma_a} \overrightarrow{g}^T + \frac{n_m}{f_m} \frac{1}{\|h\|_2} \overrightarrow{\sigma_m} \overrightarrow{m}^T \tag{40}$$

or

$$\|\sigma_{\mathbf{B}}\|_F \leq \frac{n_a}{f_a} \|\sigma_a\|_2 + \frac{n_m}{f_m} \|\sigma_m\|_2 \quad , \tag{41}$$