This file is just a bunch of random stuff and notes for myself (and others, of course).

1 ENU to NED transformations

I had the problem very often that I have to transform form ENU no NED. The simple conversion: "Flip x and y and negate z" doesn't work for quaternions or if you want to use matrix algebra.

1.1 Matrix

Flipping x and y and negating z is easy to express as a matrix:

$$R_{ENU2NED} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{1}$$

This works in both directions, since $R_{ENU2NED} = R^T_{ENU2NED}$.

1.2 Quaternion

It's easy to compute a quaternion out of the above rotation matrix.

$$q_{ENU2NED} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(i + j) \tag{2}$$

This makes sense, since the real value = 0 represents a rotation about 180° and the three values for the axis $\overrightarrow{v} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T$ represent the axis of rotation.

Transforming a quaternion between ENU/NED

If you want to cahnge a quaternion from NED to ENU or vice versa. It's not totally simple like for vectors.
If your quaternion consist of the values:

$$q_{ECEF2NED} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \tag{3}$$

Then a transformation to ENU (NED) is made as following:

$$q_{ECEF2NED} = \frac{1}{\sqrt{2}} \begin{pmatrix} -b - c \\ a + d \\ a - d \\ -b + c \end{pmatrix} \tag{4}$$

Pay attention doing it twice! The multiplication of an NED to ENU quaternion with itself leads to

$$q_{ECEF2NED} \bullet q_{ECEF2NED} \bullet \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -a \\ -b \\ -c \\ -d \end{pmatrix}. \tag{5}$$

This is logically the same rotation, but mathemaically a different quaternion. So don't be confused if all values are negative :-)

$$
\begin{tikzcd}
\text{Noise} \arrow[d, "\text{Gaussian noise}"] \\
\text{Measurement} \arrow[d, "\text{weight of a single measurement}"] \\
\text{attitude profile matrix} \arrow[d, "\text{(see the section below)}"] \\
\text{``K'' - matrix} \arrow[d, "\text{computational error}"] \\
\text{Eigenvector} \arrow[d, "\text{computational error}"] \\
\text{Euler angle}
\end{tikzcd}
$$

Fig. 1: Propagation of uncertainty

2 Initialisation

2.1 What about the standard deviation?

First of all, every sensor (accelerometers $\vec{a}$ and magnetometers $\vec{m}$) has gaussian noise, that can be expressed as an additive error:

$$\vec{a} + \vec{\sigma_a} \qquad \vec{m} + \vec{\sigma_m} \tag{6}$$

It can be asssumed that the error follows a standard deviation (has zero mean and is time-invariant). The attitude profile matrix $\mathbf{B}$ is the sum of the measurements with specific weights.

$$\mathbf{B} = \sum_{k=1}^{n} w_k \cdot \vec{W}_k \cdot \vec{V}_k^T = w_a \sum_{k=1}^{n_a} \frac{\vec{a}_k}{\|a_k\|} \cdot \vec{g}^T + w_m \sum_{k=1}^{n_m} \frac{\vec{m}_k}{\|m_k\|} \cdot \vec{h}^T \tag{7}$$

n is the number of measurements, w_k is the specific weight of a measurement, $\vec{W}_k$ the measured vector and $\vec{V}_k$ the reference direction, which belongs to the measured direction. Therefore n_a is the number of acceleration measurements, w_a is the (constant) weight of the acceleration measurements, $\vec{a}_k$ is a single acceleration observation and $\vec{g}$ is the gravity. $\vec{a}_k$ becomes normed. Similar for the magnetometer weight w_m, measurement $\vec{m}_k$, the magnetic field $\vec{h}$ and the amount of magnetometer measurements n_m. See the next section how the weight should be choosen.

The resulting error is

$$\sigma_{\mathbf{B}} = \frac{n_a}{f_a} \frac{1}{\|g\|_2} \vec{\sigma_a} \vec{g}^T + \frac{n_m}{f_m} \frac{1}{\|h\|_2} \vec{\sigma_m} \vec{m}^T \tag{8}$$

The error for the "K"-matrix is easy to get by inserting $\mathbf{B} + \sigma_{\mathbf{B}}$ into

$$\mathbf{K} = \begin{bmatrix} trace(\mathbf{B}) & \overrightarrow{Z}^T \\ \overrightarrow{Z} & \mathbf{B} + \mathbf{B}^T - trace(\mathbf{B})\mathbf{I} \end{bmatrix} \tag{9}$$

2.2 choosing the best weight for the attitude profile matrix

If you replace the single measurements in equation (7) with the real (and normed) measurements

$$\frac{\overrightarrow{a}_k + \overrightarrow{\sigma_a}}{\|a_k\|_2} \qquad \frac{\overrightarrow{m}_k + \overrightarrow{\sigma_m}}{\|m_k\|_2} \tag{10}$$

and assume that $\mathbf{B}$ has an error $\mathbf{B} + \sigma_{\mathbf{B}}$, you will get

$$\mathbf{B} + \sigma_{\mathbf{B}} = w_a \sum_{k=1}^{n_a} \frac{\overrightarrow{a}_k + \overrightarrow{\sigma_a}}{\|a_k\|_2} \cdot \overrightarrow{g}^T + w_m \sum_{k=1}^{n_m} \frac{\overrightarrow{m}_k + \overrightarrow{\sigma_m}}{\|m_k\|_2} \cdot \overrightarrow{h}^T \tag{11}$$

$$\mathbf{B} + \sigma_{\mathbf{B}} = w_a \sum_{k=1}^{n_a} \frac{\overrightarrow{a}_k}{\|a_k\|_2} \cdot \overrightarrow{g}^T + \frac{\overrightarrow{\sigma_a}}{\|a_k\|_2} \cdot \overrightarrow{g}^T + w_m \sum_{k=1}^{n_m} \frac{\overrightarrow{m}_k}{\|m_k\|_2} \cdot \overrightarrow{h}^T + \frac{\overrightarrow{\sigma_m}}{\|m_k\|_2} \cdot \overrightarrow{h}^T \tag{12}$$

$$\mathbf{B} + \sigma_{\mathbf{B}} = \underbrace{w_a \sum_{k=1}^{n_a} \frac{\overrightarrow{a}_k}{\|a_k\|_2} \cdot \overrightarrow{g}^T + w_m \sum_{k=1}^{n_m} \frac{\overrightarrow{m}_k}{\|m_k\|_2} \cdot \overrightarrow{h}^T}_{\mathbf{B}} + w_a \sum_{k=1}^{n_a} \frac{\overrightarrow{\sigma_a}}{\|a_k\|_2} \cdot \overrightarrow{g}^T + w_m \sum_{k=1}^{n_m} \frac{\overrightarrow{\sigma_m}}{\|m_k\|_2} \cdot \overrightarrow{h}^T \tag{13}$$

$$\sigma_{\mathbf{B}} = w_a \sum_{k=1}^{n_a} \frac{\overrightarrow{\sigma_a}}{\|a_k\|_2} \cdot \overrightarrow{g}^T + w_m \sum_{k=1}^{n_m} \frac{\overrightarrow{\sigma_m}}{\|m_k\|_2} \cdot \overrightarrow{h}^T \tag{14}$$

$\|a_k\|_2$ and $\|m_k\|_2$ shouldn't vary that much and can be assumed as constant ($\|a\|_2$ and $\|m\|_2$). The equation reduces to:

$$\sigma_{\mathbf{B}} = w_a n_a \frac{\overrightarrow{\sigma_a}}{\|a\|_2} \cdot \overrightarrow{g}^T + w_m n_m \frac{\overrightarrow{\sigma_m}}{\|m\|_2} \cdot \overrightarrow{h}^T \tag{15}$$

It would be nice, if it's possible to reduce this to a single value. To do that, we need a matrix norm. In this case, I choosed the Frobenius Norm:

$$\begin{aligned} \|\sigma_{\mathbf{B}}\|_F &= \|w_a n_a \frac{\overrightarrow{\sigma_a}}{\|a\|_2} \cdot \overrightarrow{g}^T + w_m n_m \frac{\overrightarrow{\sigma_m}}{\|m\|_2} \cdot \overrightarrow{h}^T\|_F & (16) \\ &\leq \|w_a n_a \frac{\overrightarrow{\sigma_a}}{\|a\|_2} \cdot \overrightarrow{g}^T\|_F + \|w_m n_m \frac{\overrightarrow{\sigma_m}}{\|m\|_2} \cdot \overrightarrow{h}^T\|_F & (17) \\ &= w_a n_a \frac{1}{\|a\|_2} \cdot \|\overrightarrow{\sigma_a}\overrightarrow{g}^T\|_F + w_m n_m \frac{1}{\|m\|_2} \cdot \|\overrightarrow{\sigma_m}\overrightarrow{h}^T\|_F & (18) \end{aligned}$$

It is straight-forward to proove that $\|\overrightarrow{a}\,\overrightarrow{b}^T\|_F = \|a\|_2 \cdot \|b\|_2$

$$\|\sigma_{\mathbf{B}}\|_F \leq w_a n_a \frac{\|g\|_2}{\|a\|_2} \cdot \|\sigma_a\|_2 + w_m n_m \frac{\|h\|_2}{\|m\|_2} \cdot \|\sigma_m\|_2 \tag{19}$$

As you can see, the uncertainty depends on the following parameters:

- The weight of a measurement w_a and w_m.
- The number of measurements n_a and n_m.

- Something that I call a ”measurement gain”, $\frac{\|a\|_2}{\|g\|_2}$ and $\frac{\|m\|_2}{\|h\|_2}$, since it’s the ratio between the true value and the measured value.
- The maximum of the error σ_a and σ_m.

This is not what I want. I don’t want the error grow with the number of measurements or with the gain, that is related to the measruement device. If I choose

$$w_a = \frac{\|a\|_2}{n_a \cdot \|g\|_2} \quad and \quad w_m = \frac{\|m\|_2}{n_m \cdot \|h\|_2} \tag{20}$$

I get something like

$$\|\sigma_{\mathbf{B}}\|_F \leq \|\sigma_a\|_2 + \|\sigma_m\|_2 \quad , \tag{21}$$

which looks much better. For the Frobenius norm of the attitude profile matrix the choosen weight leads to

$$\|\mathbf{B}\|_F \leq \frac{1}{n_a}\sum_{k=1}^{n_a} \|a_k\|_2 + \frac{1}{n_m}\sum_{k=1}^{n_m} \|m_k\|_2 \quad . \tag{22}$$

That is an acceptable fact, since it helps to keep the matrix bound. But because I want to do live-update of the attitude profile matrix I don’t know the real amount of measurements n_a and n_m. But I know the measurement frequencies f_a and f_m, which are directly linked to them ($f = \frac{n}{T}$). So my final decision for the measurement weight is

$$w_a = \frac{\|a\|_2}{f_a \cdot \|g\|_2} \quad and \quad w_m = \frac{\|m\|_2}{f_m \cdot \|h\|_2} \quad . \tag{23}$$

The resulting error is then

$$\sigma_{\mathbf{B}} = \frac{n_a}{f_a}\frac{1}{\|g\|_2}\overrightarrow{\sigma_a}\,\overrightarrow{g}^T + \frac{n_m}{f_m}\frac{1}{\|h\|_2}\overrightarrow{\sigma_m}\,\overrightarrow{m}^T \tag{24}$$

or

$$\|\sigma_{\mathbf{B}}\|_F \leq \frac{n_a}{f_a}\|\sigma_a\|_2 + \frac{n_m}{f_m}\|\sigma_m\|_2 \quad , \tag{25}$$